



EXTERNAL

Cboe® European-Style Options Implied Volatility Calculation Methodology

This document details a methodology used to construct a volatility surface from European settlement style options. This volatility surface can be used to generate intra-day and end-of-day indicative prices for existing or hypothetical option series.

Implied Volatility Calculation Method

Implied Discount Factor and Forward Price

For each maturity, we calculate the discount factor (DF) and the forward (F) by the following methodology:

Using put-call parity, we calculate the implied forward price from the European-style call and put prices around the at-the-money strike. For each maturity, we select the strikes satisfying the following criteria: ATM – 8% to ATM + 8% and perform Linear Regression.

For each maturity and each selected strike K_i $i \in \{1, 2, \dots, n\}$, we define $\text{call}(K_i)$ and $\text{put}(K_i)$ as the mid-price of the call and put options with strike K_i , respectively.

Put-call parity states that for call and put options with the same strike and maturity:

$$\text{call} - \text{put} = S - DF \times K - Div$$

where:

- S is the underlying index value;
- Div is the discrete dividend amount;

Using the put-call parity formula for each selected strike, we have the following linear model:

$$\begin{aligned} Y &= \alpha + \beta X, \\ Y_i &= \text{put}(K_i) - \text{call}(K_i) - S \\ X_i &= K_i \end{aligned}$$

We perform linear regression to fit this model. The ordinary least squares estimation gives us:

$$\begin{aligned} \beta &= \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} \\ \alpha &= \bar{Y} - \beta \bar{X} \end{aligned}$$

where: $\bar{X} = \frac{\sum X_i}{n}$, $\bar{Y} = \frac{\sum Y_i}{n}$

Given:

$$DF = \beta$$

$$Div = \alpha$$

$$Forward Price = F = \frac{S - Div}{DF}$$

As the maturity of the option, T, will not generally be a listed maturity, we must now interpolate the discount factor and forward. We select T₁ and T₂ as the closest maturity less or equal to T and the closest maturity greater to T, respectively

$$DF_T = DF_1 + \frac{DF_2 - DF_1}{T_2 - T_1} \times (T - T_1)$$

$$F_T = F_1 + \frac{F_2 - F_1}{T_2 - T_1} \times (T - T_1)$$

Calculation of Implied Volatility

The implied volatility (V) for each maturity and strike level is calculated by equating the Black-Scholes formula to the observed option price from the market and the implied forward price calculated in the section above. For each maturity,

$$|Model Price_i - Option Price_i| < Tolerance$$

For the call,

$$Model Price(F, DF, T, K, V)_i = DF_i \times (F_i \times Norm(d_1) - K_i \times Norm(d_2))$$

For the put,

$$Model Price(F, DF, T, K, V)_i = DF_i \times (K_i \times Norm(-d_2) - F_i \times Norm(-d_1))$$

where:

$$d_1 = \frac{LN\left(\frac{F}{K}\right) + \frac{V^2}{2} \times (T - t)}{T_2 - T_1}$$

$$d_2 = d_1 - V \times \sqrt{T - t}$$

And:

$$Norm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$$

Smile Interpolation and Extrapolation

To calculate the implied volatility for a non-listed strike and a non-listed maturity, we utilize two different techniques for smile interpolation and extrapolation: (i) the spline interpolation for smile interpolation and (ii) the classical SABR model for smile extrapolation.

Spline Interpolation

The spline interpolation is used for the strike range $K- \leq K \leq K+$, and the expiration range $T- \leq T \leq T+$. We use a cubic spline interpolation in total implied variance space. This is a method to interpolate unknown values between given points using piecewise third order polynomials. The idea is to fit a piecewise function of the form:

$$f(x) = \begin{cases} f_1(x) & x_1 \leq x < x_2 \\ f_2(x) & x_2 \leq x < x_3 \\ \dots & \dots \\ f_{n-1}(x) & x_{n-1} \leq x < x_n \end{cases}$$

$$x = K$$

$$f(x) = V(K)$$

For each f_i is a third-order polynomial defined by:

$$f_i(x) = \alpha_i x^3 + \beta_i x^2 + \gamma_i x + \delta_i$$

$$f_i(x_i) = \alpha_i x_i^3 + \beta_i x_i^2 + \gamma_i x_i + \delta_i = f_{i-1}(x_i)$$

Next, we make the first and second derivatives agree at each adjacent points.

$$f'_i(x) = 3\alpha_i(x - x_i)^2 + 2\beta_i(x - x_i) + \gamma_i = f'_{i-1}(x)$$

$$f''_i(x) = 6\alpha_i(x - x_i) + 2\beta_i = f''_{i-1}(x)$$

Now, we have $4(n-1)$ coefficients and $4(n-1)$ equations, therefore, from basic algebra, we could derive the coefficients, and the volatility between known implied volatility.

SABR Model

The SABR model extrapolates the smile for $0 \leq K(T) \leq K(T) -$ (left extrapolation) and $K(T) + < K(T)$ (right extrapolation).

The SABR model was introduced as a simple class of stochastic volatility processes for the underlying. Given traded and liquid options, we fit the SABR model on the observed smile and estimate the parameters. Using these parameters, we can estimate implied volatility to price at various points on the volatility surface.

The SABR model assumes that the forward rate and the instantaneous volatility are driven by two correlated Brownian motions:

$$\begin{aligned}df_t &= \alpha_t F_t^\beta dW_t^1 \\d\alpha_t &= v\alpha_t dW_t^2 \\E[dW_t^1 dW_t^2] &= \rho dt\end{aligned}$$

where:

- α is the instantaneous vol;
- v is the vol of vol;
- ρ is the correlation between the Brownian motions driving the F and the v ;
- β is the leverage effect;

The formula that the implied volatility must satisfy is:

$$|SABR \text{ Implied Vol} - \text{Observed Implied Vol}| < \text{Tolerance}$$

$$\begin{aligned}SABR \text{ Implied Vol}(T, F, K, \alpha, \beta, v, \rho) \\&= \frac{\alpha z}{\chi(z) F K^{(1-\beta)/2}} \left(1 + \frac{(1-\beta)^2 (\ln F/K)^2}{24} + \frac{(1-\beta)^4 (\ln F/K)^4}{1920} \right)^{-1} \\&\times \left(1 + \left(\frac{(1-\beta)^2 (\alpha)^2}{24(FK)^{1-\beta}} + \frac{\alpha \beta v \rho}{4(FK)^{(1-\beta)/2}} + \frac{(2-3\rho^2)v^2}{24} \right) T \right),\end{aligned}$$

Where:

$$z = \frac{v \ln F}{\alpha} (FK)^{\frac{1-\beta}{2}}, \chi(z) = \ln \left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho} \right)$$

Given the initial value of the forward, F , the model is governed by 4 parameters, α , β , v , ρ .

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